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ON NASH EQUILIBRIUM PROGRAMS OF CAPITAL ACCUMULATION UNDER ALTRUISTIC PREFERENCES*

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1. INTRODUCTION

An important problem in intertemporal economics is to examine whether intergenerational equity is compatible with Pareto-efficient allocation of resources.

There is no unanimity about what constitutes an appropriate concept of intergenerational equity. Rawls [1972] has undertaken a rather exhaustive study of the concept of distributive justice, but has some reservations against the applicability of the concept proposed by him in an atemporal context, to intertemporal economics. Dasgupta [1974a, b] has suggested that the concept of Nash noncooperative equilibrium, in an intertemporal context, corresponds to the universalizability criterion of distributive justice discussed by Rawls. We will consider this equilibrium solution to be our concept of intergenerational equity.

We consider a simple aggregative model of capital accumulation, with a preference structure that reflects altruism in the specific sense that each generation's welfare is defined not only on its own consumption level, but also on that of its immediate descendants. For such an intertemporal society, a program will be considered to be given by an infinite sequence of savings ratios. A Nash noncooperative equilibrium is a sequence of savings ratios for which the optimal choice of a savings ratio by each generation, on the assumption that all other generations save in accordance with this program, is also as prescribed by this program.

With reference to interior stationary Nash equilibria, it has been maintained by Dasgupta [1974a, b] that they are Pareto inefficient. Earlier, an identical result had been obtained by Phelps and Pollak [1948] in the context of a more complex preference structure. In both models, it was assumed that the utility function has constant elasticity and that the technology is linear.

To make a statement about the possible Pareto inefficiency of a particular program, one must compare it with all other feasible programs that emanate from the same initial conditions. The above authors take this to be the initial endowment of the economy. This is certainly the correct use of the Pareto efficiency concept if we are considering the possible Pareto efficiency of a program from the beginning of biological time. In Section 4 of this paper, we extend the existing results to include *all* interior Nash equilibria, under much *weaker* assumptions.

However, we argue in this paper, that if there is altruism reflected in the pref-

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erence structure, and *if there is always a preceding generation*, then generation one inherits not only a particular endowment of capital, but also a "moral obligation" or a "contract" (not necessarily enforceable) to save at the rate that the preceding generation supposed it would do when making its own plans. If this is not the case then generation one can hardly pre-suppose that generation two will honor their "moral obligation" to generation one. And then, at least within the confines of a non-stochastic model, the concept of a Nash equilibrium would not be the appropriate solution concept.

Our primary purpose in this paper is to show that, with this added restriction on the set of Pareto comparable feasible programs, a certain class of Nash equilibrium programs is Pareto efficient; that is, for this class, there is no conflict between intergenerational equity and Pareto efficiency. Without this added restriction, it is not surprising that all Nash equilibria are Pareto inefficient; for, "all" generations can be made better off if the "initial" generation, whose interests are no longer considered, is allowed to pay a sufficient price. In both the analysis of Dasgupta, and of Phelps-Pollak, the comparison program has this property.

The procedure followed in our paper may be briefly summarized here. First, we show that an interior program is a Nash equilibrium program if and only if it is "quasi-competitive," a term which corresponds to the notion of competitive behavior in the context of an externality which is not internalized. Second, a quasi-competitive program is shown to be Pareto efficient if an obvious transversality condition is satisfied. The last step is to consider the circumstances under which this transversality condition is satisfied. In following this route, we also note other interesting results, such as the relation between efficient and Pareto-efficient Nash equilibrium programs, and some asymptotic properties of Nash equilibria. In Sections 2–6, we include statements of all results, and verbal discussions pertaining to them. All proofs are postponed to Section 7.

2. THE MODEL

2.1. *Production.* We consider a one-good economy, with a technology given by a function, f, from R^+ to itself. The production possibilities consist of inputs x, and outputs $y=f(x)$, for $x \geq 0$. The following assumption on f is maintained throughout:

(*F*) $f(0)=0$, *and for* $x \geq 0$, f *is strictly increasing, concave and twice differentiable.*

This assumption is sometimes strengthened to the following:

(F') $f(x) = dx^{\alpha}$ *for* $x \geq 0$, *where* $d > 0$, *and* $0 < a \leq 1$.

A linear technology is obtained when (F') holds, and $a=1$.

We consider the initial input level, $\underline{x}$, to be historically given and positive. A *feasible production program* is a sequence $\langle x, y \rangle = \langle x_t, y_{t+1} \rangle$ satisfying

(1) $x_0 = \underline{x}, 0 \leq x_t \leq y_t$ for $t \geq 1, f(x_t) = y_{t+1}$ for $t \geq 0$.

The *consumption program* $\langle c \rangle = \langle c_t \rangle$ generated by $\langle x, y \rangle$ is

(2) $c_t = y_t - x_t \ (\geq 0)$, for $t \geq 1$.

The sequence $\langle x, y, c \rangle$ is called a *feasible program*, it being understood that $\langle x, y \rangle$ is a production program, and $\langle c \rangle$ the corresponding consumption program. A feasible program $\langle x, y, c \rangle$ is called *interior* if $c_t > 0$ for $t \geq 1$.

A feasible program $\langle x, y, c \rangle$ is *inefficient* if there is a feasible program $\langle x', y', c' \rangle$, such that $c'_t \geq c_t$ for $t \geq 1$, and $c'_t > c_t$ for some t. It is *efficient* if it is not inefficient.

The *consumption-ratio sequence* $\langle z \rangle = \langle z_t \rangle$ associated with a feasible program $\langle x, y, c \rangle$ is given, for $t \geq 1$, by

(3) $z_t = \frac{c_t}{y_t}$ if $y_t > 0$, $z_t = 0$ if $y_t = 0$.

The corresponding *saving-ratio sequence* is $\langle s \rangle = \langle s_t \rangle = \langle 1 - z_t \rangle$. A feasible program $\langle x, y, c \rangle$ is *stationary* if $z_t = z_{t+1}$ for $t \geq 1$.

2.2. *Preferences.* Individuals are considered to be identical except for their dates of birth. The group of individuals born at the beginning of period t is called the t-th generation. Each generation lives for precisely one period, and is replaced by an equal number of direct descendants the instant they die.

The preferences of each generation are the same, and are representable by a welfare function U from $R^+ \times R^+$ to R^*. We consider generation t's welfare, denoted by u_t, to be dependent on its own consumption, and on the consumption of its immediate descendants. Thus, we can associate with a feasible program $\langle x, y, c \rangle$, a welfare sequence $\langle u \rangle = \langle u_t \rangle$, given by

(4) $u_t = U(c_t, c_{t+1})$ for $t \geq 1$.

The following assumption on U is maintained throughout:

(U) *$U(c, c') = v(c) + bv(c')$, for $(c, c') \geq 0$, where v is a function from R^+ to R^*, and $b > 0$.*

We will refer to v as a *utility function*. We assume that v satisfies

(V) *$v(.)$ is increasing, concave and twice differentiable* for $c > 0$.

This assumption is sometimes strengthened to the following:

(V') *$v(.)$ is twice differentiable, and $[v''(c)c/v'(c)] = w(c) = w$, for $c \geq 0$, where $0 > w > -\infty$.*

(V') implies that v is representable in the form $v(c) = -c^{-\hat{e}}$[if $\hat{e} > 0$, so $w = -(1 + \hat{e}) < -1$]: $v(c) = c^{-\hat{e}}$[if $0 > \hat{e} > -1$, so $-1 < w = -(1 + \hat{e}) < 0$]; $v(c) = \log c$(so $w = -1$). The additive and multiplicative constants can be ignored, for the set of Nash equilibria (which is the solution concept discussed below) is invariant under linear increasing transformations of the utility function, $v(.)$.

In choosing to look at the economy from $t=1$ onwards, and taking $x_0 = \underline{x}$ as historically given, we are not presupposing that $t=0$ represents the beginning of biological time. There is always a preceding generation, and generation one's choice of c_1 affects the welfare of generation zero. It is not necessary to consider the history of the economy further back in time than $t=0$, for altruism is assumed to extend only to one's immediate descendants.

Our formulation of preferences indicates that a generation's altruism ($b>0$) does not extend, even indirectly, over *all* future generations, as would be the case if $u_t = U(c_t, u_{t+1})$, for example. Even if our concern is assumed to extend over any finite number of generations, we doubt that this is a very restrictive assumption. Also, altruism is effective only insofar as one's immediate descendants choose to consume the output bequeathed to them.

2.3. *Concept of Nash Equilibrium.* We assume that each generation can choose its own consumption ratio. Equivalently, each generation can choose a consumption schedule $c_t(y)$ but it must be linear with zero intercept.

We wish to consider a solution concept which will locate those feasible programs which, in some minimal sense, imply a "just" savings principle and so an equitable allocation among generations. To find such a savings principle, Rawls proposed the hypothetical construct of the "original position." All generations, past, present and future, belong to the original position at any time t; however, each generation is assumed ignorant as to when it will exist. A member of the original position is required to assume it is born at some arbitrary time, t, and to ask if $\bar{s}_t$ is the best savings ratio from its point of view, on the assurance that all past and future generations live up to their obligations to save according to the sequence $\langle \bar{s} \rangle$. A prerequisite for $\langle \bar{s} \rangle$ to be considered "just" is that this requirement of individual rationality be met for all generations; furthermore, as t is arbitrary, there would be an internal contradiction if such programs were not intertemporally consistent.

These considerations are captured in the Nash equilibrium solution concept, i.e., for all t, $\bar{z}_t$ must maximize generation t's utility subject to all other consumption ratios being given by $\langle \bar{z} \rangle$. Formally, a feasible program $\langle \bar{x}, \bar{y}, \bar{c} \rangle$ is a *Nash equilibrium program* if its consumption ratio sequence $\langle \bar{z} \rangle$ is such that for $t \geq 1$, and $0 \leq z \leq 1$,

$$U[\bar{z}_t \bar{y}_t, \bar{z}_{t+1} f((1-\bar{z}_t)\bar{y}_t)] \geq U[z\bar{y}_t, \bar{z}_{t+1} f((1-z)\bar{y}_t)]. \tag{5}$$

(See Dasgupta [1974a, b] or Lane [1977] for a more complete discussion.)

Given any solution concept, such as the Nash equilibrium concept, a natural question to ask is whether the set of Nash equilibria is non-empty. Since it is well-known that under fairly general conditions, Nash equilibrium programs exist (see, for example, Peleg and Yaari [1973]) we shall not devote any space to this question. Rather, we shall take for granted that Nash equilibrium programs exist, and study some qualitative and normative properties of such programs.

We will restrict our attention to Nash equilibria which are interior. This eliminates programs for which x_t or z_t becomes zero for some date; these programs can hardly be considered "just" in a world in which the existence of a next generation is never an issue. Thus, our restriction does not limit seriously the scope of the exercise.

3. PRELIMINARY PROPERTIES OF NASH EQUILIBRIUM PROGRAMS

In this section, we study some elementary properties of Nash equilibrium programs, and establish existence and uniqueness results relating to stationary Nash equilibrium programs.

First, we note that if $\langle x, y, c\rangle$ is an interior Nash equilibrium

(6) $$v'(c_t) = bv'(c_{t+1})z_{t+1}f'(x_t) \qquad \text{for} \quad t \geq 1.$$

At the margin, generation t is indifferent between consuming an extra unit of output and bequeathing it to its immediate descendants who would then increase their consumption by $z_{t+1}f'(x_t)$.

It is useful to introduce the differentiable mapping, g, such that

(7) $$z_t = g(z_{t+1}, y_t).$$

iff (6) holds. This defines the set N_x of consumption ratio sequences $\langle z\rangle$, which correspond to interior Nash equilibria, that can be generated from $x_0 = x$. Let N be the subset of N_x for which every member is independent of $x_0 = x$.

In view of the fact that the generations are all alike, a Nash equilibrium program of particular interest is a stationary one. Let $\bar{N}$ be the subset of N, for which every member satisfies $z_t = z_{t+1}$ for $t \geq 1$. By (7), they satisfy $z = g(z, y_t)$ for $t \geq 1$, for some choice of $z > 0$.

The following properties of g are useful and readily verified.

LEMMA 1. *Under (F), (U), (V), g satisfies the following conditions*

(8) $$\operatorname{sign} \frac{\partial g}{\partial z_{t+1}} = -\operatorname{sign}\left[w(c_{t+1}) + 1\right].$$

(9) $$\operatorname{sign} \frac{\partial g}{\partial y_t} = \operatorname{sign}\left[w(c_t) - w(c_{t+1})\frac{x_t f'(x_t)}{f(x_t)} - \frac{x_t f''(x_t)}{f'(x_t)}\right].$$

It is, of course, clear from (9) that under (F′), (U), (V′), sign $\partial g/\partial y_t = \operatorname{sign}(1-a)\cdot(1+w)$. Using these properties of g, we have the following uniqueness result regarding stationary Nash equilibrium programs:

PROPOSITION 1. *Under (F), (U), (V), the set $\bar{N}$ has at most one member if either* (i) $\inf_{x\geq 0} f'(x) < 1 < \sup_{x\geq 0} f'(x)$, *or* (ii) $w(c) = w$, *and* $[f'(x)x/f(x)] = 1$.

It remains to prove an existence result regarding stationary Nash equilibrium programs; that is, to find conditions under which $\bar{N}$ is non-empty. Consider the graph of g in the (z_{t+1}, z_t) space, given y_t. The graph will in general shift with a change in y_t. Suppose a member $\langle z \rangle$ belongs to $\bar{N}$, and $\langle y \rangle$ is the associated output sequence. Then, $g(z, y_t) = g(z, y_{t+1})$. Note that this is certainly satisfied if g is independent of y_t, so the mapping does not shift. So, in particular, the initial input of the economy does not affect the sequence $\langle z \rangle$, and $N = N_x$. It is also satisfied if y_t is a constant and equal to $f(x_0)$ in which case the mapping may shift, but it will pivot around the point (z, z). Suppose (F'), (U), (V') are satisfied. Then in the former case, (9) implies $a=1$, or $w=-1$. In the latter case, constancy of z and y implies constancy of c and x as well. So (6) implies that $x^{1-a} = (dab)z$. But also, $dx^a = c + x = zdx^a + x$. The simultaneous solution yields $x_0^{1-a} = x^{1-a} = [dab/(1+ab)]$, and the initial input of the economy is all important. This suggests

PROPOSITION 2. *Under (F'), (U), (V'), an interior stationary Nash equilibrium program exists iff at least one of the following three conditions is satisfied*: (i) $w=-1$, (ii) $a=1$, (iii) $\underline{x}^{1-a} = dab/(1+ab)$.

The following example shows that, under quite reasonable conditions, a stationary Nash equilibrium program may fail to exist. Let $f(x) = 2x^{1/2}$, $v(c) = c^{1/2}$, $\underline{x}=1$, $b=1/2$. Then, clearly $w \neq -1$, and $a \neq 1$. Also, $\underline{x}^{1-a}/d = 1/2 > 1/4 \{1+(1/4)\} = ab/(1+ab)$. Hence, by Proposition 2, no interior stationary Nash equilibrium exists. It is also clear from Proposition 2 that unless the technology is linear, the existence of a stationary Nash equilibrium is an exception, rather than the rule. So, in the next two sections, we extend the scope of our discussion and results beyond the confines imposed by stationarity.

Dasgupta [1974a, b] restricts his attention to the strengthening of assumptions (F'), (V'), given by $a=1$, and $w<-1$. Then (9) indicates that the mapping g

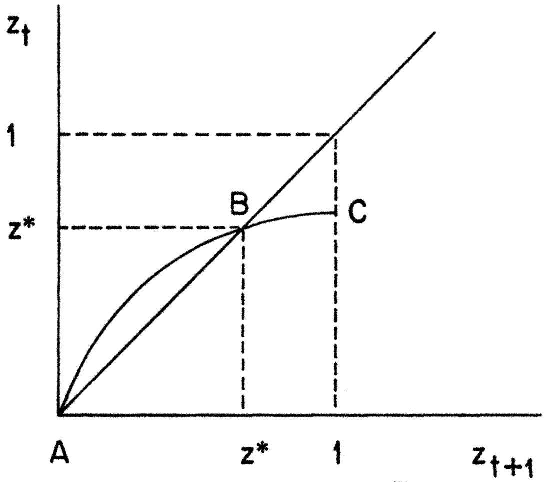

FIGURE 1: $a=1$, $w<-1$; $N_x = N$, $\bar{N} = \{\langle z^* \rangle\}$.

does not shift with changes in y_t; so $N_x = N$, and the set N can easily be illustrated, as in Figure 1. By (8), as $w < -1$, so $\partial g/\partial z_{t+1} > 0$, and there are two stationary equilibria, at A and B, but only the latter is interior. Hence, $\bar{N}$ consists of the single member $\langle z^* \rangle$.

It is a consequence of the assumption $w < -1$, that $\langle z^* \rangle$ is "unstable." If $z_1 > z^*$, then z_t is strictly increasing, and the economy will exhaust the stock of the commodity in finite time, leaving nothing for subsequent generations to consume. This situation is hardly "just." If $z_1 < z^*$, then z_t is strictly decreasing, and it converges to zero fast enough to make the sequence $\langle z_t \rangle$ summable, and this, in turn, implies inefficiency of the corresponding Nash equilibrium program. For these reasons, Dasgupta only considers the normative significance of interior stationary Nash equilibria.

However, if $w > -1$, then AC slopes downwards. So $\langle z^* \rangle$ is stable, and the consumption ratio sequences $\langle z \rangle$ that approach $\langle z^* \rangle$ should be considered. If $w = -1$, then AC is horizontal, and (6) has only one solution, namely $z_t = 1/(1+b)$, that is $\bar{N} = N$. Even if $a < 1$, the same result is observed, with $z_t = 1/(1+ab)$. In general, of course, AC will shift and then not only is it unlikely that a stationary Nash equilibrium exists, but also cyclical behavior in the consumption ratio sequence $\langle z \rangle$ may be observed.

4. ON THE CONCEPT OF PARETO EFFICIENCY

In this section, we discuss the concept of Pareto efficiency that is used in the literature, and propose an alternative definition that is more appropriate for the solution concept we are studying.

Under the assumptions $a = 1$, and $w < -1$, Dasgupta maintains that the unique stationary Nash equilibrium is Pareto inefficient. It is worthwhile to clarify the meaning of this result. However, first his result is extended to include all interior Nash equilibrium programs, under a much weaker set of assumptions.

PROPOSITION 3. *Under (F), (U), (V), if $\langle \bar{x}, \bar{y}, \bar{c} \rangle$ is any interior Nash equilibrium program, then there is a feasible program $\langle x, y, c \rangle$ such that $c_1 < \bar{c}_1$, and $u_t > \bar{u}_t$ for all $t \geq 1$.*

Under Dasgupta's assumptions, this same result has been observed by Phelps and Pollak [1968], although they base their analysis on a considerably more complex preference structure, where the welfare of all descendants enters the welfare function of generation t. The customary interpretation of a result like Proposition 3 is that the program $\langle \bar{x}, \bar{y}, \bar{c} \rangle$ is Pareto inferior to the program $\langle x, y, c \rangle$. This is certainly the correct interpretation if $t = 0$ is regarded as the beginning of biological time.

The static version of the Prisoner's Dilemma is often cited as a reason for expecting this type of result. Perhaps, though, this intertemporal model is more akin to the infinitely repeated game formulation of the Prisoner's Dilemma.

Then, there need be no conflict between a Nash equilibrium and Pareto efficiency.

If we consider that for every generation, there is a preceding generation, and we are looking at the economy from $t=0$ onwards, then our choice of c_1 would affect the generation born in $t=0$. From Proposition 3, as $c_1<\bar{c}_1$, so clearly $u_0<\bar{u}_0$, and so generation zero has been made worse off. We cannot then conclude Pareto inefficiency if we include the interests of generation zero in our ordering of programs. Generation zero anticipates that generation one will consume more than they in fact consume, with the result that generation zero chooses a consumption level which, ex post, is revealed non-optimal. Of course, it is obvious that the program $\langle \bar{x}, \bar{y}, \bar{c} \rangle$ can be dominated from $t=1$ onwards, if the preceding generation is always allowed to pay an appropriate price, objections incurred based on our assumed "non-overlapping" of generations notwithstanding.

If generation one can behave in this manner towards generation zero then, by the same token, it must anticipate that generation two may do the same to them. Then, the Nash equilibrium concept does not imply rationality at the level of the individual.

There are at least two ways out of this dilemma. One is to consider a stochastic model in which each generation makes a decision on the basis of a probability distribution, defined on the next generation's consumption level, and centered around its expected level. The other method, and the one discussed in this paper, is to suppose that generation one inherits an enforceable contract, or a moral obligation to generation zero to consume the anticipated amount. It will not be disputed that two feasible programs are Pareto comparable only if the initial "state" of the economic system is the same for both programs. The state is a summary of the past behavior of the system, insofar as it is relevant for decisions to be made today. So what seems to have been overlooked is that the state is summarized not only by the inherited level of output, but also by the inherited moral obligation to consume the anticipated amount. To the extent that the economic system is inherited, and not subject to choice, the set of Pareto comparable programs is smaller, and the possiblity of conflict between the concepts of Pareto efficiency and that of a Nash equilibrium is reduced.

Even if $t=0$ is regarded as the beginning of biological time, these considerations are still relevant if one is interested in Pareto efficiency from $t=2$ onwards: for, then, there is a preceding generation and an inherited moral obligation.

This result is also implied by the earlier Rawlsian interpretation, at least if the original position at any moment of time included all past generations. For then generation zero belongs to the original position at $t=0$, *and* $t=1$, so the inherited contract would have to be honored: otherwise its consent to a program extending from $t=1$ would not be consistent with its choice of a program extending from $t=0$.

The above observations indicate that we should treat c_1 as historically given, just as we consider the initial input $\underline{x}$ to be historically given. So we shall consider a feasible program $\langle x, y, c \rangle$ as Pareto comparable with a feasible program

$\langle x', y', c' \rangle$ if $c_1 = c'_1$.

DEFINITION 1. A feasible program $\langle x, y, c \rangle$ is *Pareto inefficient* if there is a feasible program $\langle x', y', c' \rangle$ such that (i) $c'_1 = c_1$; (ii) $u'_t \geq u_t$ for $t \geq 1$, and $u'_t > u_t$ for some t. A feasible program is called *Pareto efficient* if it is not Pareto inefficient.

Our solution concept is that of a non-cooperative Nash equilibrium in which every generation is endowed with the right to choose its own savings ratio. The appeal of a non-cooperative solution concept in welfare economics is really that it captures some aspects of a "rights" based political philosophy when binding agreements cannot necessarily be assumed. There are some things that individuals should be allowed to choose in light of their own self-interest even if the overall outcome reduces social welfare. Our analysis is an enquiry as to what is an appropriate endowment of rights, or zone of control, for each individual, so that there is no conflict between a "rights" based political philosophy and efficiency. It is a result of this enquiry that there will be no such conflict if the zone of control is constrained as indicated in part (i) of Definition 1.

This definition also enables us to identify *uniquely* the source of the Phelps-Pollak and Dasgupta "inefficiency" as summarized in Proposition 3. For if an interior Nash equilibrium is Pareto-efficient, then the *only* program which yields an improvement in welfare for the set of generations born after $t = 0$ is one which reduces the welfare of generation zero. Furthermore, this program could not be considered "just," for it would not receive the consent of generation zero which is a member of the original position at $t = 1$. Therefore, the restriction in Definition 1, that $c'_1 = c_1$ (or equivalently $c'_1 \geq c_1$) is both *natural* and *minimal* if we wish to give a Rawlsian interpretation to our results.

It should be noted that whether or not generation zero plays an active role in the decision making process at $t = 1$ is not pertinent insofar as we wish to consider whether the program from $t = 1$ is "just" in the Rawlsian sense. In fact, our analysis is based on the supposition that generation zero is not alive when the decision on c_1 is made. It is true that the model can be reformulated as one of "overlapping" generations with each generation living for two periods and each with a preference structure defined only over its own consumption levels in the two periods. There are no considerations of altruism. If we assume that at each moment of time the "young" choose the savings ratio, and that output (net of savings) is distributed equally (or in some pre-determined proportion) between themselves and their parents, then the results below will not be modeified in any essential way. In this alternative formulation, even though generation zero is alive when the decision on c_1 is made, it still plays no role in the decision making process at that time.

However, the essential point is that considerations of *justice* require us to determine a program that *would* receive the consent of all parties, just *as if* they were actively involved in the decision making process. The "original position" is simply a hypothetical concept that enables one to capture this notion.

5. NORMATIVE PROPERTIES OF NASH EQUILIBRIUM PROGRAMS

In this section we will isolate the class of Nash equilibrium programs which are Pareto efficient in the sense of Definition 1.

To this end, we first show that Nash equilibrium programs behave quasi-competitively. The meaning of this term is given in:

DEFINITION 2. An interior program $\langle \bar{x}, \bar{y}, \bar{c} \rangle$ is *quasi-competitive* if there is an associated price sequence $\langle \bar{q}, \bar{p} \rangle = \langle \bar{q}_t, \bar{p}_t \rangle$, such that $\bar{q}_t > 0$, and $\bar{p}_t > 0$, for $t \geq 0$, and

$$\text{(10)} \quad \bar{q}_t \bar{u}_t - \bar{p}_t \bar{c}_t - (\bar{p}_{t+1}/\bar{z}_{t+1})\bar{c}_{t+1} \geq \bar{q}_t U(c, c') - \bar{p}_t c - (\bar{p}_{t+1}/\bar{z}_{t+1})c' \quad \text{for} \quad c, c' \geq 0, t \geq 1.$$

$$\text{(11)} \quad \bar{p}_{t+1}\bar{y}_{t+1} - \bar{p}_t \bar{x}_t \geq \bar{p}_{t+1} f(x) - \bar{p}_t x, \quad \text{for} \quad x \geq 0, t \geq 0.$$

Let $\langle x, y, c \rangle$ be any feasible program. Generation t must allocate output y_t between c_t and x_t; these are variables under direct control. To do this optimally, it must anticipate a given level of z_{t+1}, since $u_t = v(c_t) + bv(c_{t+1}) = v(c_t) + bv(z_{t+1} \cdot f(x_t))$. That is, the assumed level of z_{t+1} constitutes an externality, which has not been internalized in the preference structure.

If this optimal choice corresponds to that implied by the program $\langle \bar{x}, \bar{y}, \bar{c} \rangle$, it follows that we must take prices as $\langle \bar{q}, \bar{p} \rangle$ and consider the externality to be fixed at $\bar{z}_{t+1}$, so that the two programs are comparable. Therefore, the imputed value of generation t's inherited output is $\bar{p}_t c_t + \bar{p}_{t+1} y_{t+1} = \bar{p}_t c_t + (\bar{p}_{t+1}/\bar{z}_{t+1}) c_{t+1}$, where $\bar{p}_{t+1}$ is the futures price of a unit of output at $(t+1)$, and therefore $(\bar{p}_{t+1}/\bar{z}_{t+1})$ is the futures price of a unit of consumption at $(t+1)$. It then follows that condition (10) is to be interpreted as "utility maximization subject to a budget constraint" in the presence of an externality which has not been internalized. Condition (11) constitutes "intertemporal profit maximization" and there is no externality present here. For these reasons, we call the above conditions quasi-competitive.

Conditions (10) and (11) are similar to those obtained in the literature on optimal and efficient growth theory (see Gale and Sutherland [1968], Majumdar-Mitra-McFadden [1976]), except that the marginal rate of transformation between c_{t+1} and y_{t+1} is unity in these models. That is, the futures price of a unit of consumption, and a unit of output, are the same. In the model discussed here, in contrast, the marginal rate of transformation between c_{t+1} and y_{t+1} is z_{t+1}.

The marginal rate of substitution between y_{t+1} and c_t, given z_{t+1}, is $[v'(c_t)/bv'(c_{t+1})z_{t+1}]$. The marginal rate of transformation is $f'(x_t)$. In a quasi-competitive equilibrium the two are equal, and yield the interior Nash equilibrium condition (6). The common slope of the indifference curve and the production possibility curve provide the price ratio implicit in (10) and (11). This suggests:

PROPOSITION 4. *Under (F), (U), (V), an interior program $\langle \bar{x}, \bar{y}, \bar{c} \rangle$ is a*

Nash equilibrium program, iff it is quasi-competitive.

In view of the differentiability assumptions on f and v, the prices $\langle q, p \rangle$ associated with a quasi-competitive equilibrium $\langle x, y, c \rangle$ are determined uniquely (up to a positive scalar multiple), and given by: $p_0 = 1, p_{t+1} = p_t/f'(x_t)$ for $t \geq 0$; $q_0 = 1$, $q_t = p_t/v'(c_t)$ for $t \geq 1$. Hence, when we refer to prices associated with any Nash equilibrium, we will be referring precisely to the above-defined prices.

We would now like to find an easily applicable criterion that can test the Pareto efficiency of Nash equilibrium programs. The following proposition provides us with such a criterion.

PROPOSITION 5. *Under (F), (U), (V), if an interior Nash equilibrium program $\langle x, y, c \rangle$ satisfies*

$$\inf_{t \geq 0} p_t x_t = 0 \tag{12}$$

then it is Pareto-efficient.

We refer to (12) as the *transversality condition*. This condition is a rather strong sufficient condition for Pareto efficiency. Thus, a Pareto-efficient Nash equilibrium need not satisfy (12), even if it is stationary (see example below). It is therefore worthwhile to look for weaker conditions which ensure Pareto efficiency of a Nash equilibrium program. If we restrict our attention to stationary Nash equilibria, and replace (F) by (F′), then the necessary and sufficient conditions of Pareto-inefficiency are given by:

PROPOSITION 6. *Under (F′), (U), (V), a stationary interior Nash equilibrium program is Pareto-inefficient if and only if*

$$\inf_{t \geq 0} p_t x_t > 0, \quad \textit{and} \quad \sum_{t=0}^{\infty} \frac{1}{p_t x_t} < \infty. \tag{13}$$

The following example indicates that a Nash equilibrium could be Pareto efficient, but violate (12). Let $f(x) = 2x^{1/2}$, $b = 2$. The program $\langle x, y, c \rangle$ given by $x_t = c_t = 1$, for $t \geq 1$, $y_t = 2$ for $t \geq 1$, is feasible from $x_0 = \underline{x} = 1$. It satisfies (6), and is a Nash equilibrium program. Since $p_t x_t = 1$ for $t \geq 0$, so it is Pareto efficient by Proposition 6. However, it clearly violates (12).

Under (F′), (13) is the necessary and sufficient condition for a stationary program to be inefficient (see Corollary 5 in Mitra [1979]). Hence, we have the following interesting result:

COROLLARY 1. *Under (F′), (U), (V), a stationary interior Nash equilibrium program $\langle x, y, c \rangle$ is Pareto-efficient iff it is efficient.*

It should be mentioned that much weaker conditions than (F′) can be used to obtain Corollary 1, following the methods used in Mitra [1979, Theorem 1]. We do not attempt such generalizations here, as the proof of Proposition 6 indicates how such extensions can be achieved, and because such results are silent on the Pareto-efficiency of non-stationary Nash equilibria.

It would be interesting to obtain results which parallel Proposition 6 and Corollary 1, for all interior Nash equilibrium programs. One way is to restrict the model further to ensure that (12) becomes both necessary and sufficient for Pareto efficiency of Nash equilibria. This means, in turn, ruling out cases demonstrated by our above example. This is easily accomplished if we restrict b to be ≤ 1; that is, we assume that generations care for themselves at least as much as they care for their children. With this additional restriction, we have

PROPOSITION 7. *Under* (F'), (U), (V), *and* $b \leq 1$, *an interior Nash equilibrium program* $\langle x, y, c\rangle$ *is Pareto efficient iff* (12) *holds.*

Using Proposition 7, the following result is then evident:

COROLLARY 2. *Under* (F'), (U), (V), *and* $b \leq 1$, *an interior Nash equilibrium program* $\langle x, y, c\rangle$ *is Pareto efficient iff it is efficient.*

Corollary 2 says that if the prices are "right" as far as intertemporal *allocation of resources* is concerned (the quasi-competitive program is efficient), then they cannot be "errant" in regard to achieving a Pareto efficient *distribution of goods* among generations (the quasi-competivive program is Pareto efficient).

Just as Proposition 4 provides a price characterization of a Nash equilibrium program, so the following result presents a price characterization of Pareto efficient Nash equilibrium programs.

PROPOSITION 8. *Under* (F'), (U), (V), *and* $b \leq 1$, *an interior program* $\langle x, y, c\rangle$ *is a Pareto efficient Nash equilibrium program iff* (i) *it is quasi-competitive, and* (ii) *it satisfies* (12).

Since the consumption-ratio figures as a basic variable in all of our analysis, it is worthwhile to obtain a restatement of Proposition 7, in which the transversality condition (12) is replaced by a condition on z_t. For this purpose, we find it useful to separate the case of a non-linear technology ($a<1$) from the case of a linear technology ($a=1$), as the behavior of z_t in the two cases is not quite the same. We then have the following two results:

PROPOSITION 9. *Under* (F'), (U), (V) $b \leq 1$, *and* $a=1$, *an interior Nash equilibrium program is Pareto efficient iff*

$$\lim_{T\to\infty} \sum_{t=1}^{T} z_t = \infty . \tag{14}$$

PROPOSITION 10. *Under* (F'), (U), (V), $b \leq 1$, *and* $a<1$, *an interior Nash equilibrium program is Pareto efficient iff*

$$\limsup_{t\to\infty} z_t > 0. \tag{15}$$

Two remarks are in order regarding the above results. First, it is clear that if an interior quasi-competitive program satisfies (15), then it is Pareto-efficient, regardless of whether the technology is linear or non-linear. Since (15) says that the consumption-ratio be bounded away from zero for some subsequence of periods, we have demonstrated that interior Nash equilibria are Pareto-efficient under very weak restrictions. This should be contrasted with Proposition 3 where we showed that every interior Nash equilibrium was "Pareto inefficient" in the sense used by Dasgupta.

Second, it is easy to establish that if $a=1$, and $d>1$ (the linear technology is productive), then an interior Nash equilibrium program is Pareto efficient iff (15) holds. For unproductive linear technologies, such a claim cannot be made.

We finally come to the existence result regarding Pareto-efficient Nash equilibria. This is accomplished by using Proposition 2, and the normative properties of Nash equilibria proved in this section. More precisely, the stationary Nash equilibria, which are shown to exist (by Proposition 2) can be verified to be Pareto efficient, if we have $b(1-a)\leq 1$. This gives us the following theorem.

THEOREM 1. *Under* (F'), (U), (V'), $b(1-a)\leq 1$, *there exists an interior stationary Pareto-efficient Nash equilibrium iff one of the following three conditions is satisfied*: (i) $w=-1$, (ii) $a=1$, (iii) $\underline{x}^{1-a}=dab/(1+ab)$.

A more satisfactory existence result would be one which was not restricted to stationary programs. Under (F'), (U), (V'), if the technology is non-linear $(a<1)$, $w\neq -1$, and $\underline{x}^{1-a}\neq dab/(1+ab)$, there could exist non-stationary Pareto-efficient Nash equilibria, although by Theorem 1, there could not exist stationary ones. We finally note:

THEOREM 2. *Under* (F'), (U), (V'), *there exists a Pareto-efficient Nash equilibrium if one of the following four conditions is satisfied*: (i) $w=-1$, *and* $b(1-a)\leq 1$, (ii) $a=1$, (iii) $\underline{x}^{1-a}=dab/(1+ab)$, *and* $b(1-a)\leq 1$, (iv) $w>-1$, *and* $b\leq 1$.

6. SOME CONCLUDING COMMENTS

In this section we would like to relate the results of our paper to those of two earlier papers in the literature in this area, namely the contributions of Phelps-Pollak [1968], and Kohlberg [1976]. The welfare of generation t in the Phelps-Pollak formulation is:

$$u_t = v(c_t) + \delta \sum_{\tau=1}^{\infty} \alpha^\tau v(c_{t+\tau}) = v(c_t) + \alpha u_{t+1} + \alpha(\delta - 1)v(c_{t+1})$$

If $\delta=1$, then

$$u_t = v(c_t) + \alpha u_{t+1} = \sum_{\tau=0}^{\infty} \alpha^\tau v(c_{t+\tau}).$$

Thus, any program that increases c_t will increase u_{t-1}. Therefore the conflict of interests discussed earlier does not arise and, in accordance with intuition and well known results in control theory, any interior Nash equilibrium program will be Pareto efficient; the restriction $c'_1 = c_1$ given in Definition 1 plays no role.

It is natural to define *perfect altruism* to correspond to the absence of any conflict of interest, that is $\delta = 1$. It is as if there is complete cooperation among generations because we have attributed to them the appropriate *ethical values* which induce them to behave in this manner. However, Rawls argues ([1972], p. 584) that we should avoid attributing any ethical motivations to generations; we do not ask what a generation's preference function "should be," but rather we let them make decisions on the basis of their "selfish" interests insofar as they can ascertain them.

Now while a generation's selfish interests may extend to its children and perhaps to their children also, it is an empirical question whether they extend into the affairs of distant future generations. These considerations are in accordance with assuming only a limited form of altruism, as is the case if either $\delta \neq 1$ (following Phelps-Pollak) or $u_t = v(c_t) + bv(c_{t+1})$ (following Dasgupta). Both formulations embody the same inherent conflict of interests; Pareto efficiency in the usual sense is lost.

Our results identify uniquely the source of this inefficiency. Pareto efficiency can be restored if the condition of Definition 1, that is, the "inherited moral obligation," is satisfied. This indicates, in a precise way, the need for a form of limited collective behavior, or social contract, that will in effect make this "moral obligation" a binding agreement.

In an overlapping generations framework perhaps the social contract could take the form of a system of automatic adjustments in the levels of social security taxes (paid by the young), and pensions (received by the old), by which all generations must abide. The rate of tax paid by generation $t+1$ could be a predetermined function, known by generation t at time t, of the pension paid by generation t to generation $t-1$. One presumes that there exists a function which would modify behavior based on individual rationality so that there is no conflict with the principles of intergenerational justice; that is, that condition (i) of Definition 1 is always satisfied. However, this is offered only as a suggestion and is an appropriate area for further research.

Kohlberg [1976], using a model which is a special case of the one studied in this paper, and others (for example, Loury [1976]) in the context of somewhat different models, have enlarged the strategy space to allow each generation to choose a consumption schedule, $c_t(y)$, which may be non-linear. However, all generations are required to choose the same consumption schedule. Although this restriction might appear "natural" since all generations have the same preferences, and face the same technology, there is no demonstration of the fact that along a Nash equilibrium, the consumption schedules would *have* to be identical. If they are not, then the additional conditions which force them to be identical are worth studying. [In this paper, we have restricted consumption schedules to be linear, but have not restricted them to be the same for all generations]. The conditions under which different consumption schedules are forced to be linear also merit additional research.

If the consumption schedule can be non-linear, then along a Nash equilibrium, the marginal rate of substitution between c_t and y_{t+1} depends on the marginal propensity to consume $c'(y_{t+1})$, rather than the average propensity, z_{t+1}. This is the essential difference implied by the solution concept of Kohlberg, and that used here. Consequently, the size of the strategy space matters, and the results are not quite comparable, unless the consumption schedule in Kohlberg's exercise turns out to be linear. Kohlberg provides some conditions under which the consumption schedule will be linear, and under these conditions, the results coincide. Kohlberg does not examine whether his equilibrium consumption schedule generates programs which are Pareto-efficient. This also remains an interesting open question, the answer to which would further illuminate the relation between his concept of equilibrium and ours.

7. PROOFS

PROOF OF LEMMA 1. Substitute $c_{t+1}=z_{t+1}f(x_t)$, $c_t=z_ty_t$ and $x_t=(1-z_t)y_t$ in (6). Write $z_t=g(z_{t+1}, y_t)$, and partially differentiate the equation with respect to z_{t+1} and y_t. The relations (8) and (9) are obtained by using (F) and (V).

PROOF OF PROPOSITION 1. Suppose $\langle z\rangle$ and $\langle \bar{z}\rangle$ belong to $\bar{N}$, and $z\neq\bar{z}$. Call the corresponding Nash equilibrium programs $\langle x, y, c\rangle$ and $\langle \bar{x}, \bar{y}, \bar{c}\rangle$ respectively.

Under condition (i) of the proposition, there exist unique numbers x, $\bar{x}$, satisfying $(1-z)f(x)=x$, and $(1-\bar{z})f(\bar{x})=\bar{x}$. Clearly, $x_t\to x$ and $\bar{x}_t\to\bar{x}$, as $t\to\infty$. Hence, $c_t\to zf(x)=c$, and $\bar{c}_t\to\bar{z}f(\bar{x})=\bar{c}$, as $t\to\infty$. By continuity of v' and f', we have $v'(c)=bv'(c)zf'(x)$ and $v'(\bar{c})=bv'(\bar{c})\bar{z}f'(\bar{x})$. Hence $zf'(x)=(1/b)=\bar{z}f'(\bar{x})$. Without loss of generality, suppose $z>\bar{z}$; then, since $(1-z)[f(x)/x]=(1-\bar{z})[f(\bar{x})/\bar{x}]$, so $\bar{x}\geq x$, and $f'(x)\geq f'(\bar{x})$. Hence $zf'(x)>\bar{z}f'(\bar{x})$, which is a contradiction.

Under condition (ii) of the proposition, (6) implies that $z(1-z)^w=1/[bd^{(w+1)}]=\bar{z}(1-\bar{z})^w$. Let $h(k)=k(1-k)^w$. Then, $h(k)$ is continuous on $[0, 1)$, $h(0)=0$, $h(k)\to\infty$, as $k\to 1$, and $h(k)$ is increasing. Hence, there is a unique solution, $\hat{k}$, to the equation $h(k)=1/[bd^{(w+1)}]$, with $0<\hat{k}<1$. This contradiction establishes the result.

PROOF OF PROPOSITION 2. (*Necessity*) Suppose there exists an interior stationary Nash equilibrium program; call it $\langle x, y, c\rangle$. Let the stationary consumption-ratio associated with it be z. Then, by (6),

$$v'(zy_t) = bv'(zy_{t+1})zf'(x_t). \tag{16}$$

Clearly, we must have either (a) $w=-1$, or (b) $w\neq -1$. In case (b), (16) reduces to

$$y_t^w = b\ y_{t+1}^w z\ d\ a\ x_t^{a-1}. \tag{17}$$

Using the facts that $x_t = (1-z)y_t$, and $y_{t+1} = f(x_t)$, in (17), we have

$$\frac{(1-z)^{-w}}{z} = abd^{1+w}x_t^{(1+w)(a-1)}. \tag{18}$$

The left-hand side depends only on z, and, hence, is a constant. Since $w \neq -1$, we must, therefore, have either $(b')a = 1$, or $(b'')a \neq 1$, and x_t is a constant equal to $x_0 = \underline{x}$. In case (b''), since $c_t + \underline{x} = f(\underline{x}) = d\underline{x}^a$, we must have, upon simplification

$$(1-z) = \frac{\underline{x}^{1-a}}{d}. \tag{19}$$

Combining (19) with (18), we have $\underline{x}^{1-a} = dab/(1+ab)$.

(*Sufficiency*) We note first that if an interior program satisfies (6) then, by concavity of f and v, it is a Nash equilibrium program. Thus, all we have to show is that under each of the three conditions of the proposition, there is a stationary interior program satisfying (6). We shall consider the three cases in turn.

If case (i) holds, define a program $\langle x, y, c\rangle$ in the following way: let $x_0 = \underline{x}$, $y_{t+1} = f(x_t)$, for $t \geq 0$; $x_t = y_t - c_t$, and $c_t = \hat{z}y_t$, where $\hat{z} = 1/(1+ab)$ for $t \geq 1$. Then, clearly, $\langle x, y, c\rangle$ is feasible, interior, and stationary. To check that it is a Nash equilibrium program, note that $[\hat{z}/(1-\hat{z})] = 1/ab$, so by (F'),

$$b[x_t f'(x_t)/f(x_t)]\,[\hat{z}/(1-\hat{z})] = 1. \tag{20}$$

Using the facts that $f(x_t) = y_{t+1}$, and $y_t = x_t/(1-\hat{z})$, we have

$$b\left(\frac{1}{\hat{z}y_{t+1}}\right)\hat{z}f'(x_t) = \frac{1}{\hat{z}y_t}. \tag{21}$$

Since $w = -1$, this means that (6) is satisfied, as required.

If case (ii) holds, define $\langle x, y, c\rangle$ exactly as in case (i), except that we let $\hat{z}$ be the solution of

$$h(z) = z(1 - z^w) = \frac{1}{bd^{(w+1)}}. \tag{22}$$

Now $h(z)$ is continuous on $[0, 1)$, $h(0) = 0$, $h(z) \to \infty$, as $z \to 1$, and $h(z)$ is increasing. This implies that there is a unique solution, $\hat{z}$, to (22), such that $0 < \hat{z} < 1$. Hence $\langle x, y, c\rangle$ is feasible, interior and stationary. To check that it is a Nash equilibrium program, observe that, by the definition of $\hat{z}$, we have

$$\frac{c_t^w}{d^w(1-\hat{z})^w y_t^w} = \frac{bc_{t+1}^w \hat{z}d}{y_{t+1}^w}. \tag{23}$$

Using the fact that $y_{t+1} = dx_t = d(1-\hat{z})y_t$ in (23), we obtain

$$c_t^w = bc_{t+1}^w \hat{z}d \tag{24}$$

which implies that (6) holds, as required.

If case (iii) holds, then define $\langle x, y, c\rangle$ as in case (i), except that we define $\hat{z} = 1 - (\underline{x}^{1-a}/d)$. By definition of $\underline{x}$, we know that $0 < \hat{z} < 1$, so that $\langle x, y, c\rangle$

is feasible, interior and stationary. To show that it is a Nash equilibrium program, note that the definition of $\hat{z}$ implies that (18) is satisfied, and so (17) is satisfied. Hence, (6) is satisfied, as required.

PROOF OF PROPOSITION 3. Choose θ, such that $0<\theta<\bar{c}_1$. Define a program $\langle x(\theta), y(\theta), c(\theta)\rangle$ by $x_0(\theta)=\bar{x}_0=\underline{x}$; $x_1(\theta)=\bar{x}_1+\theta$, and $x_t(\theta)=\bar{x}_t$ for $t\geq 2$. Clearly, this is a feasible program for each θ. Now, observe that, $u_1(\theta)-\bar{u}_1=v'(k_1)(c_1(\theta)-\bar{c}_1)+bv'(k_2)(c_2(\theta)-\bar{c}_2)$, where $c_1(\theta)\leq k_1\leq \bar{c}_1$, and $\bar{c}_2\leq k_2\leq c_2(\theta)$. Hence, $u_1(\theta)-\bar{u}_1=v'(k_1)(-\theta)+bv'(k_2)[f(\bar{x}_1+\theta)-f(\bar{x}_1)]=-v'(k_1)\theta+bv'(k_2)\cdot f'(r_1)\theta$ where $\bar{x}_1\leq r_1\leq \bar{x}_1+\theta$. Hence, $u_1(\theta)-\bar{u}_1=[bv'(k_2)f'(r_1)(1-\bar{z}_2)]\theta+[b\bar{z}_2v'(k_2)f'(r_1)-v'(k_1)]\theta=[e_1(\theta)+e_2(\theta)]\theta$, where $e_1(\theta)=bv'(k_2)f'(r_1)(1-\bar{z}_2)$, and $e_2(\theta)=[bv'(k_2)f'(r_1)\bar{z}_2-v'(k_1)]$. Denote $[bv'(\bar{c}_2)f'(\bar{x}_1)(1-\bar{z}_2)]$ by e. Then, as $\theta\to 0$, we have $e_2(\theta)\to[b\bar{z}_2v'(\bar{c}_2)f'(\bar{x}_1)-v'(\bar{c}_1)]=0$, since $\langle \bar{x}, \bar{y}, \bar{c}\rangle$ is an interior Nash equilibrium program. Also, $e_1(\theta)\to e$, as $\theta\to 0$. Hence, there is $\theta^*>0$, such that for $\theta\leq\theta^*$, $[e_2(\theta)]\leq(1/4)e$, and $e_1(\theta)\geq(1/2)e$. So, $u_1(\theta^*)-\bar{u}_1\geq\theta^*((1/2)e-(1/4)e)\geq(1/4)e\theta^*$. So, for the feasible program $\langle x(\theta^*), y(\theta^*), c(\theta^*)\rangle$, $c_1(\theta^*)\langle\bar{c}_1$, $u_t(\theta^*)\rangle\bar{u}_t$ for $t=1, 2$; $u_t(\theta^*)=\bar{u}_t$ for $t\geq 3$. It is trivial, now, to construct a feasible program $\langle x, y, c\rangle$ for which $c_1<\bar{c}_1$, and $u_t>\bar{u}_t$ for $t\geq 1$.

PROOF OF PROPOSITION 4. (*Necessity*) To find a price sequence satisfying (10) and (11), we will simply define $\langle\bar{q}, \bar{p}\rangle$, and check that, under the definition, conditions (10), (11) are indeed satisfied. Define

(25) $$\bar{p}_0=1,\ \bar{p}_{t+1}=\frac{\bar{p}_t}{f'(\bar{x}_t)} \qquad \text{for} \quad t\geq 0$$

(26) $$\bar{q}_0=1,\ \bar{q}_t=\frac{\bar{p}_t}{v'(\bar{c}_t)} \qquad \text{for} \quad t\geq 1.$$

Note, then, that $\bar{q}_t>0$, and $\bar{p}_t>0$, for $t\geq 0$, since $\langle \bar{x}, \bar{y}, \bar{c}\rangle$ is interior, and f', $v'>0$.

To check that (11) holds, use the concavity of f, and write, for any $x\geq 0$, $f(x)-f(\bar{x}_t)\leq f'(\bar{x}_t)(x-\bar{x}_t)$. Multiply through by $\bar{p}_{t+1}$, and use (25) to get $\bar{p}_{t+1}(f(x)-f(\bar{x}_t))\leq\bar{p}_t(x-\bar{x}_t)$. Then, (11) is obtained by transposition of the relevant terms.

To check that (10) holds, use the concavity of v, and write, for any $c, c'\geq 0$, $U(c, c')-U(\bar{c}_t, \bar{c}_{t+1})\leq v'(\bar{c}_t)(c-\bar{c}_t)+bv'(\bar{c}_{t+1})(c'-\bar{c}_{t+1})$. Since $\langle \bar{x}, \bar{y}, \bar{c}\rangle$ is an interior Nash equilibrium program, so (6) holds. Multiply the previous inequality by $\bar{q}_t$, and use (6), (25) and (26) to obtain $\bar{q}_t[U(c, c')-U(\bar{c}_t, \bar{c}_{t+1})]\leq\bar{p}_t(c-\bar{c}_t)+(\bar{p}_{t+1}/\bar{z}_{t+1})(c'-\bar{c}_{t+1})$. Then, (10) is obtained by transposition of the relevant terms.

(*Sufficiency*) Suppose that $\langle \bar{x}, \bar{y}, \bar{c}\rangle$ is quasi-competitive, but not a Nash equilibrium program. Then there is some $z(0\leq z\leq 1)$, and some time period (call it t), such that (5) is violated. Using (10), we have $\bar{p}_t z\bar{y}_t+(\bar{p}_{t+1}/\bar{z}_{t+1})\cdot\bar{z}_{t+1}f((1-z)\bar{y}_t)>\bar{p}_t\bar{z}_t\bar{y}_t+(\bar{p}_{t+1}/\bar{z}_{t+1})\bar{z}_{t+1}f((1-\bar{z}_t)\bar{y}_t)$. Let us denote $(1-z)\bar{y}_t$ by x; then, $z\bar{y}_t=\bar{y}_t-x$, and substituting this into the previous inequality, we have $\bar{p}_{t+1}f(x)+\bar{p}_t(\bar{y}_t-x)>\bar{p}_{t+1}\bar{y}_{t+1}+\bar{p}_t\bar{c}_t$. Since $\bar{c}_t=\bar{y}_t-\bar{x}_t$, we have, by transposition, $\bar{p}_{t+1}f(x)-\bar{p}_t x>\bar{p}_{t+1}\bar{y}_{t+1}-\bar{p}_t\bar{x}_t$, which violates (11), a contradiction.

PROOF OF PROPOSITION 5. Suppose $\langle x, y, c\rangle$ is an interior Nash equilibrium program, which is Pareto-inefficient. Then there is a feasible program $\langle x', y', c'\rangle$, such that $c_1'=c_1$, $u_t'\geq u_t$ for $t\geq 1$, and $u_t'>u_t$ for some t. Let τ be the first period for which $u_\tau'>u_\tau$. Then $(x_t, c_t)=(x_t', c_t')$ for $1\leq t\leq\tau$, and $x_{\tau+1}'<x_{\tau+1}$, $c_{\tau+1}'>c_{\tau+1}$. Let $u_\tau'-u_\tau=e_1$. Now, note that for $t\geq 1$, we have $p_t(c_t'-c_t)+(p_{t+1}/z_{t+1})(c_{t+1}'-c_{t+1})\geq 0$, so that it follows that

$$(27)\qquad p_t[f(x_{t-1}')-f(x_{t-1})-(x_t'-x_t)] + \frac{p_{t+1}}{z_{t+1}}[f(x_t')-f(x_t)-(x_{t+1}'-x_{t+1})]\geq 0\,.$$

Using concavity of f, and (25), and denoting $p_t(x_t-x_t')$ by d_t, we have $(d_t-d_{t-1})+(1/z_{t+1})(d_{t+1}-d_t)\geq 0$, so that

$$(28)\qquad d_{t+1}\geq d_t(1-z_{t+1})+d_{t-1}z_{t+1}\qquad \text{for}\quad t\geq 1.$$

As we have noted earlier, $d_\tau=0$, while $d_{\tau+1}>0$. Using (28), we have $d_{\tau+2}\geq d_{\tau+1}(1-z_{\tau+2})=e_2$, say. Then, using (28) again, $d_{\tau+3}\geq e_2(1-z_{\tau+3})+d_{\tau+1}z_{\tau+3}\geq e_2(1-z_{\tau+3})+e_2z_{\tau+3}=e_2$. Now, it is clear from (28) that for all $t\geq\tau+3$, we have $d_t\geq e_2$. Consequently, $p_tx_t\geq e_2$ for $t\geq\tau+3$. This implies, since $\langle x, y, c\rangle$ is interior, that $\inf_{t\geq 0} p_tx_t>0$.

PROOF OF PROPOSITION 6. (*Necessity*) Suppose $\langle x, y, c\rangle$ is an interior stationary Nash equilibrium program, which is Pareto-inefficient. Then there is a feasible program $\langle x', y', c'\rangle$, such that $c_1'=c_1$, $u_t'\geq u_t$ for $t\geq 1$, and $u_t'>u_t$ for some t. Let the stationary consumption-ratio of $\langle x, y, c\rangle$ be z. Note, first, that $a\neq 1$. For if $a=1$, then $x_{t+1}=(1-z)dx_t$, and so $p_{t+1}x_{t+1}=(1-z)p_tx_t$, so that (12) holds, and $\langle x, y, c\rangle$ is Pareto-efficient by Proposition 5. Hence, $a<1$. Now, using the method and notation of the proof of Proposition 5, we have $d_t=p_t(x_t-x_t')\geq e_2>0$, for $t\geq\tau+3$. Then, using Taylor's expansion up to the second-order, for f, in (27), we get for $t\geq\tau+4$, $[(d_{t+1}/z)+d_t]\geq[(d_t/z)+d_{t-1}]+(1/2)p_t\cdot[-f''(k_{t-1})](x_{t-1}-x_{t-1}')^2+(1/2)p_{t+1}[-f''(k_t)][(x_t-x_t')^2/z]$, where $x_{t-1}'\leq k_{t-1}\leq x_{t-1}$, and $x_t'\leq k_t\leq x_t$. It is easy to check that $p_{t+1}[-f''(k_t)](x_t-x_t')^2\geq(1-a)\cdot(d_t^2/p_tx_t)$, and this yields the following inequality:

$$[(d_{t+1}/z)+d_t]\geq[(d_t/z)+d_{t-1}] + [(1/2)(1-a)d_{t-1}^2/p_{t-1}x_{t-1}]+[(1/2)(1-a)d_t^2/p_tx_tz]\,.$$

Writing $r_{t+1}=(d_{t+1}/z)+d_t$, and letting $m'=\min(1, az/(1-z))$,

$$(29)\qquad r_{t+1}\geq r_t+\left[\frac{1}{2}\,\frac{m'(1-a)}{p_tx_t}\right][(d_t/z)^2+d_{t-1}^2]\,.$$

Since for any two real numbers, A, B, we have $(A^2+B^2)\geq(1/2)(A+B)^2$, so we have $r_{t+1} \geq r_t + [(1/4)m'(1-a)/p_t x_t]r_t^2$. Letting $m = \min(1, (1/4)m'(1-a))$ and $\bar{r}_t = mr_t$, we finally obtain

(30) $$\hat{r}_{t+1} \geq \hat{r}_t\left(1 + \left(\frac{\hat{r}_t}{p_t x_t}\right)\right).$$

Now, following the proof of Theorem 1 in Mitra [1979], we obtain

(31) $$\sum_{t=1}^{\infty} \frac{1}{p_t x_t + \hat{r}_t} < \infty.$$

Since $\hat{r}_t \leq p_t x_t[(1/z)+(1/(1-z))]$, so (31) implies (13).

(*Sufficiency*) If (13) holds, then by Corollary 5 in Mitra [1979], $\langle x, y, c\rangle$ is inefficient, and hence Pareto-inefficient.

PROOF OF COROLLARY 1. If the Nash equilibrium program $\langle x, y, c\rangle$ is Pareto-efficient, then clearly it is efficient. Conversely, if it is Pareto-inefficient, then by Proposition 6, condition (13) holds, and by Corollary 5 in Mitra, it is inefficient.

PROOF OF PROPOSITION 7. (*Sufficiency*) This follows from Proposition 5. (*Necessity*) Suppose $\langle x, y, c\rangle$ is an interior Nash equilibrium program which is Pareto-efficient. We separate two cases: (i) $a=1$, (ii) $a<1$. In case (i), we note that $\langle x, y, c\rangle$ is efficient, and so by Theorem 5 in Mitra [1979], (12) holds. In case (ii), the analysis is somewhat more complicated. We claim, first, that

(32) $$f'(x_t) \leq 1, \quad \text{implies that} \quad x_t < x_{t-1}.$$

Suppose, contrary to the claim, that $f'(x_t) \leq 1$, but $x_t \geq x_{t-1}$. Then by (6), $v'(c_t) < v'(c_{t+1})$, so $c_t > c_{t+1}$; that is $f(x_{t-1}) - x_t > f(x_t) - x_{t+1}$, so that using $x_t \geq x_{t-1}$, we have $x_{t+1} > x_t$, and $f'(x_{t+1}) \leq f'(x_t) \leq 1$. Thus the argument above can be repeated for all succeeding periods to obtain x_t monotonically increasing. Hence, there is $e>0$, such that $x_t \geq \hat{x}+e$ from a certain time onwards (where $f'(\hat{x})=1$; i.e., $\hat{x}$ is the "golden-rule" input level). Hence $\langle x, y, c\rangle$ is inefficient, a contradiction. This establishes (32).

Next, we claim that there is some t for which $x_t < \hat{x}$. If not, then $x_t \geq \hat{x}$ for all t. So, $f'(x_t) \leq 1$ for all t, and by (32), $x_t < x_{t-1}$ for all t. Hence x_t converges to $x \geq \hat{x}$. Clearly $x > \hat{x}$ is ruled out since this implies that $\langle x, y, c\rangle$ is inefficient. So $x = \hat{x}$, and so c_t converges to $\hat{c} = f(\hat{x}) - \hat{x}$, and z_t converges to $\hat{z} = 1 - (\hat{x}/f(\hat{x}))$. By continuity of v' and f', and (6), we have $b\hat{z}=1$. But $b \leq 1$, and $0 < \hat{z} < 1$, so $b\hat{z} < 1$, a contradiction, which establishes the claim.

Let t_1 be the first period for which $x_t < \hat{x}$. Then, clearly, $x_t < \hat{x}$, for $t > t_1$, as well. If not, consider the first period, $t_2 > t_1$ for which $x_{t_2} \geq \hat{x}$. Then, $x_{t_2-1} < \hat{x}$, and so by (32), $f'(x_{t_2}) > 1$, a contradiction. Hence $x_t < \hat{x}$, for $t \geq t_1$. Hence, p_t is decreasing for $t > t_1$. We claim that p_t converges to zero. If not, then p_t converges to some positive number. This implies that $f'(x_t)$ converges to 1, and x_t converges to $\hat{x}$. Then c_t converges to $\hat{c} = f(\hat{x}) - \hat{x}$, and z_t converges to $\hat{z} = 1 - (\hat{x}/f(\hat{x}))$. By continuity of v' and f', and (6), we have $b\hat{z}=1$. But, $b \leq 1$, and $0 < \hat{z} < 1$, so $b\hat{z} < 1$, a contradiction. Hence p_t converges to zero. Since x_t

is bounded above by $\hat{x}$, from a certain time onwards, so (12) holds.

(*Sufficiency*) This follows directly from Proposition 5.

PROOF OF COROLLARY 2. (*Necessity*) If a Nash equilibrium program is Pareto-efficient, it is clearly efficient, since v is increasing.

(*Sufficiency*) If an interior Nash equilibrium is efficient, then by the *proof* of Proposition 7 (which uses only (6) and efficiency of the program), we know that (12) is satisfied. Hence the program is Pareto-efficient, by Proposition 5.

PROOF OF PROPOSITION 8. (*Sufficiency*) If (i) holds, then by Proposition 4, the program is a Nash equilibrium. If (ii) holds, then by Proposition 5, it is Pareto-efficient.

(*Necessity*) If $\langle x, y, c\rangle$ is an interior Pareto-efficient Nash equilibrium program, then it satisfies (i) by Proposition 4, and it satisfies (ii) by Proposition 7.

PROOF OF PROPOSITION 9. (*Necessity*) If the interior Nash equilibrium program $\langle x, y, c\rangle$ is Pareto-efficient, then it is efficient, and so, by Lemma 2 in Mitra [1979], (14) holds.

(*Sufficiency*) If (14) holds, then we claim that (12) must hold. To see this, note that, since $a=1$, the sequence $\langle p_t c_t\rangle$ is summable. If (12) were violated, then $\inf_{t\geq 1} p_t y_t > 0$. So, the sequence $\langle z_t\rangle$ is summable, violating (14). This establishes the claim. Now, by Proposition 5, the Nash equilibrium is Pareto-efficient.

PROOF OF PROPOSITION 10. (*Necessity*) If the interior Nash equilibrium $\langle x, y, c\rangle$ is Pareto-efficient, then by Proposition 7, (12) holds. Now, $x_{t+1} = (1-z_{t+1})f(x_t) = a^{-1}(1-z_{t+1})f'(x_t)x_t$, so we have $p_{t+1}x_{t+1} = (1-z_{t+1})a^{-1}p_t x_t$. Suppose (15) is violated. Then, z_t converges to zero, and so for t large, $(1-z_{t+1})a^{-1} > 1$. This implies that $p_t x_t$ is increasing for t large, which violates (12). This contradiction establishes necessity.

(*Sufficiency*) Suppose the interior Nash equilibrium program $\langle x, y, c\rangle$ is Pareto-inefficient, but (15) holds. We will show that this leads to a contradiction. We claim, first, that (32) holds. Otherwise, if for some τ, $f'(x_\tau)\leq 1$, and $x_\tau \geq x_{\tau-1}$, then following the proof of Proposition 7, x_t is monotonically increasing for $t\geq\tau$. Since x_t is bounded above by $\bar{k} = \max(\underline{x}, \tilde{k})$, (where $f(\tilde{k}) = \tilde{k}$, i.e., $\tilde{k}$ is the maximum sustainable input level) so x_t converges to $x \geq \hat{x}$ (where $f'(\hat{x}) = 1$). So, c_t converges to $c = f(x) - x$, and z_t converges to $z = 1 - (x/f(x))$. Clearly, $x > \tilde{k}$ is ruled out, since this implies that $c < 0$. If $x < \tilde{k}$, then $c > 0$, so by continuity of v' and f', and (6), we have $bzf'(x) = 1$. But $b \leq 1$, $0 < z < 1$, and $f'(x) \leq 1$, so $bzf'(x) < 1$, a contradiction. If $x = \tilde{k}$, then $c = 0$, and $z = 0$, so z_t converges to zero, a contradiction to (15). Thus the claim (32) is established.

Next, we claim that there is some t for which $x_t < \hat{x}$. If not, then $x_t \geq \hat{x}$, and $f'(x_t) \leq 1$ for all t; hence, by (32), $x_t < x_{t-1}$ for all t. Since $x_t \geq \hat{x}$, so x_t converges to $x \geq \hat{x}$. Following exactly the argument in the previous paragraph, we then contradict (15). Thus, $x_t < \hat{x}$, for some t. Let the first period for which $x_t < \hat{x}$

be called t_1. Then, clearly, for $t \geq t_1$, $x_t < \hat{x}$. Otherwise, there is some $t > t_1$, for which $x_t \geq \hat{x}$. Let the first period for which this happens be called t_2. Then $x_{t_2} \geq \hat{x}$, while $x_{t_2-1} < \hat{x}$, so by (32), $f'(x_{t_2}) > 1$, which is a contradiction. Hence $x_t < \hat{x}$ for $t \geq t_1$, and so p_t is decreasing for $t > t_1$. Consequently, $p_t x_t$ is bounded above. But since $\langle x, y, c \rangle$ is Pareto-inefficient, so by Corollary 2, it is inefficient. Hence, by Theorem 4 in Mitra [1979], $(1/p_t x_t)$ is summable, so $p_t x_t$ is unbounded above. This contradiction establishes sufficiency.

PROOF OF THEOREM 1. (*Necessity*) If there exists an interior Nash equilibrium program, which is stationary, then by Proposition 2, we know that (i) or (ii) or (iii) must hold.

(*Sufficiency*) Under (i) or (ii) or (iii), we know that there exist interior, stationary Nash equilibrium programs, by Proposition 2. In fact we constructed such programs in the proof of Proposition 2. Hence, if we can check that those constructed programs are Pareto-efficient, then we will have established sufficiency. We consider the three cases in turn. If case (i) holds, then using (6), we have $y_{t+1} = bzf'(x_t)y_t$. Using the fact that $x_t = (1-z_t)y_t$, we have $x_{t+1} = bzf'(x_t)x_t$. Hence, $p_{t+1}x_{t+1} = bzp_t x_t$. In the constructed program of Proposition 2, $z = 1/(1+ab)$. Now, since $b(1-a) \leq 1$, so $bz = b/(1+ab) \leq 1$. Hence, $p_t x_t$ is bounded above, and so by Proposition 6, it is Pareto efficient.

If case (ii) holds, then $y_{t+1} = dx_t$, and $x_{t+1} = (1-z)y_{t+1}$, so $x_{t+1} = (1-z)dx_t$. It follows that $p_{t+1}x_{t+1} = (1-z)p_t x_t$, so $p_t x_t$ converges to zero. By Proposition 5, the program is Pareto-efficient.

If case (iii) holds, and $a \neq 1$, then $a < 1$, and there is a "golden rule" input level, $\hat{x} = (ad)^{1/(1-a)}$. By definition of $\underline{x}$, we know that $\underline{x} \leq \hat{x}$. Hence, p_t is bounded above, and so $p_t x_t = p_t \underline{x}$ is bounded, too. By Proposition 6, the program is Pareto-efficient.

PROOF OF THEOREM 2. If cases (i) or (ii) or (iii) hold, then the result follows from Theorem 1. Thus, we are left with the case where $w > -1$, $a < 1$, $b \leq 1$. In this case, by Peleg and Yaari [1973, Theorem 7.1], there exists a Nash equilibrium program, $\langle x, y, c \rangle$. We will show that $\langle x, y, c \rangle$ is Pareto-efficient. We claim first that

$$(33) \qquad z_\tau > 0 \quad \text{implies} \quad 0 < z_t < 1, \quad \text{for} \quad t < \tau.$$

To verify this, note that since $z_\tau > 0$, so $y_\tau > 0$, and $x_t > 0$ for $t < \tau$. This means that $z_t < 1$ for $t < \tau$. So, we only have to check that $z_t > 0$ for $t < \tau$. Suppose $z_t = 0$, for some $t < \tau$. Let the last period for which this is true be called t_1. Then $z_{t_1} = 0$, and $z_{t_1+1} > 0$. Hence, $c_{t_1} = 0$, and $c_{t_1+1} > 0$. But this would violate (5), since $v'(0) = \infty$. Thus, the claim (33) is established.

Next, we distinguish between two cases; (1) $z_t = 1$ for some t; (2) $z_t < 1$, for all t. In case (1), let t_2 be the first period for which $z_t = 1$. Then, for $t < t_2$, $0 < z_t < 1$, by (33). Hence, the Nash equilibrium is interior for $0 < t < t_2$. Then, we could define (q_t, p_t), as in (25), (26), for $0 \leq t < t_2$, and show that the Nash equilibrium is quasi-competitive at these prices, for $0 \leq t < t_2$. We claim that the program $\langle x, y, c \rangle$ is Pareto-efficient. If not, then there is a program $\langle x', y', c' \rangle$ such that $c'_1 = c_1$, $u'_t \geq u_t$ for all t, and $u'_t > u_t$ for some t. Consider t_3 to be the first period for which $u'_t > u_t$. Clearly, $t_3 < t_2 - 1$. Then, following the proof of Proposition 5, but using strict concavity of f, we obtain (28) with strict inequality for $t_3 < t < t_2$. This implies that $x'_{t_2} < x_{t_2}$. But $x_{t_2} = 0$ since $z_{t_2} = 1$, so $x'_{t_2} < 0$, violating feasibility of $\langle x', y', c' \rangle$. Hence, in case (1), there exists a Pareto-efficient Nash equilibrium.

In case (2), we claim that $0 < z_t < 1$, for all t. If not, then let the first period for which $z_t = 0$, be called t_4. Since $\underline{x} > 0$, so by (5), $z_1 > 0$; hence $t_4 > 1$. Then, we have $z_{t_4 - 1} > 0$, $z_{t_4} = 0$. Since $0 < z_t < 1$, for $t = t_4 - 1$, so $y_{t_4} > 0$, and $c_{t_4} = 0$. But this would violate (5), since $v'(0) = \infty$. Hence, in case (2), $0 < z_t < 1$, for all t; that is, $\langle x, y, c \rangle$ is an interior program. We claim that this program is Pareto-efficient. If not, then by Proposition 10, z_t converges to zero. Note that by (6), we obtain

$$x_t = abc_t^{-w} c_{t+1}^{(1+w)}. \tag{34}$$

Rearranging (34) yields the following equation:

$$x_t^{1-a} = (ab)^{1/(1+w)} d z_{t+1} [z_t/(1 - z_t)]. \tag{35}$$

Since z_t converges to zero, so x_t converges to zero, by (35). Hence p_t converges to zero, and so does $p_t x_t$. Then, by Proposition 5, $\langle x, y, c \rangle$ is Pareto-efficient. This contradiction proves the claim that $\langle x, y, c \rangle$ is Pareto-efficient, and completes the proof in case (iv).

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